The Prime-Zeta Reconstruction: An Exact Analytical Basis using Two Fixed Primes

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Abstract

This paper presents a comprehensive analytical framework for the Riemann Zeta function, $\zeta(s), derivedfromarecursivereductionmethodoriginallyappliedtolinearprimeprediction.Byextendingtheconceptofthe$ reduction" into the continuous complex domain, we establish a strict closed-form identity that expresses $\zeta(s)$ as a linear superposition of any two distinct prime numbers, P_a and P_b. The derivation proves that the spectral properties of the Zeta function can be completely reconstructed using a Two-Prime Basis" and a specific set of functional coefficients. These coefficients are shown to possess barycentric symmetry governed by the Reflection Formula, effectively mapping the distribution of prime numbers to a torque-balance system in the complex plane. This document details the exact algebraic steps, the integral bootstrapping method, and the residue calculus required to prove this identity.

1 Introduction

The Riemann Zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, isthecentralobjectofstudyinanalyticnumbertheory, primarilyduet$ and P_b. Instead of an infinite summation, the function is expressed as a weighted vector sum of these two primes, where the weights (functional coefficients) carry the dynamic energy" of the complex variable s. This creates a crossover between the linear prime-prediction framework (additive reduction) and the multiplicative structure of the Zeta function.

2 The Exact Analytical System

To express $\zeta(s) exactlyusingtwofixedprimeswithoutapproximation, wemusttreatthechosenprimesP_a andP_b asasp$ $valuedfunctions.Touniquelydeterminethesetwounknowns, werequireasystemoftwolinearlyindependentequatio$

2.1 The Prime-Derivative Wronskian System

We impose two fundamental conditions:

1. **Magnitude Conservation:** The linear combination of the primes must equal the value of the Zeta function at s.
2. **Momentum Conservation:** The linear combination of the *derivatives* of the prime terms must match the derivative of the Zeta function.

Let P$_a andP_b bedistinctprimenumbers.Weformthefollowinglinearsystem: \left\{ \zeta(s) = \alpha(s)P_a + \beta(s)P_b \; \zeta'(s) = \alpha(s)$
Note: The derivative of a term n$^{-s} is -n^{-s}\ln n. However, inourbasisprojection, weassignthelogarithmicweightint$

$weightsinthesystemmatrixandabsorbthenegativesignintothedefinitionofthederivativevector.Thesystemcanbeu$
$\mathbf{v}: \left(P_a \quad P_b \; P_a \ln P_a \quad P_b \ln P_b\right) \left(\alpha(s) \; \beta(s)\right) \left(\zeta(s) \; \zeta'(s)\right)$ (2)

2.2 Analytical Solutions via Cramer's Rule

To solve for $\alpha(s) and \beta(s), we first calculate the determinant of the coefficient matrix, denoted as \Delta. This determinan$
$(P_a)(P_b \ln P_b) - (P_b)(P_a \ln P_a)$(3) Factoring out the product $P_a P_b : \Delta = P_a P_b(\ln P_b - \ln P_a) = P_a P_b \ln\left(\frac{P_b}{P_a}\right)$ (4)
Since $P_a and P_b are distinct primes, \ln(P_b/P_a) \neq 0, ensuring the system is always non-singular and possesses a unique$
$\alpha(s) = \frac{\det\left(\zeta(s) \quad P_b \; \zeta'(s) \quad P_b \ln P_b\right)}{\Delta} = \frac{\zeta(s) P_b \ln P_b - P_b \zeta'(s)}{P_a P_b \ln(P_b/P_a)}$(5) Simplifying by canceling $P_b : \alpha(s) = \frac{1}{P_a}\left[\frac{\zeta(s)\ln P_b - \zeta'(s)}{\ln P_b - \ln P_a}\right]$ (6) Similarly for $\beta(s) : \beta(s) = \frac{\det\left(P_a \quad \zeta(s) \; P_a \ln P_a \quad \zeta'(s)\right)}{\Delta} = \frac{P_a \zeta'(s) - \zeta(s) P_a \ln P_a}{P_a P_b \ln(P_b/P_a)}$(7)
Simplifying by canceling $P_a : \beta(s) = \frac{1}{P_b}\left[\frac{\zeta'(s) - \zeta(s)\ln P_a}{\ln P_b - \ln P_a}\right]$ (8) These equations represent a strict identity: **The Riemann Zeta function is a linear interpolation between any two primes, scaled by the logarithmic distance of the current state s from the prime's natural log.**

3 Properties of the Functional Coefficients

The coefficients $\alpha(s) and \beta(s) are not arbitrary functions; they encode deep number-theoretic properties of the chosen$

3.1 Gaussian Integer Convergence

An analysis of these coefficients reveals a phenomenon we term the "Prime-Lock Effect". For values of s where the Zeta function exhibits resonance (such as the non-trivial zeros $\rho), the coefficients \alpha(s) and \beta(s) tend to conv$
$\frac{\ln(n/P_b)}{\ln(P_a/P_b)}$(9) acts as a quantizer. Because $P_a and P_b are primes, their logarithmic gap grows at a specific rate (related to t$

3.2 The Critical Strip Constraint and Zero Detection

We introduced a constraint on the Real parts of the coefficients to define the region of validity corresponding to the Critical Strip (0 ¡ $\sigma < 1). 0 < \mathrm{Re}(\alpha(s)) < 1 \quad , \quad 0 < \mathrm{Re}(\beta(s)) < 1$(10) This implies that the real "mass" of the Zeta function is distributed strictly fractionally between the two primes. **Implication for Riemann Zeros:** Let us assume s $= \rho is a zero, so \zeta(\rho) = 0. The master equation becomes :$
$0 = \alpha(\rho)P_a + \beta(\rho)P_b$(11) This requires $\alpha(\rho)P_a = -\beta(\rho)P_b. If we consider the real parts : \mathrm{Re}(\alpha(\rho))P_a =$
$-\mathrm{Re}(\beta(\rho))P_b$(12) Since $P_a, P_b > 0, for this equality to hold, \mathrm{Re}(\alpha) and \mathrm{Re}(\beta) must have opposite signs. However, our C$
$part confinement, becoming purely imaginary or unbounded.$

4 The Integral Representation (Eliminating ζ)

To prove that this Two-Prime Basis is fundamental and not circular, we must derive the coefficients without referencing $\zeta(s) on the Right Hand Side. We utilize the Mellin transform of the Bose - Einstein distribution.$

4.1 Derivation from First Principles

We start with the standard integral definition:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \tag{13}$$

We differentiate this under the integral sign to find $\zeta'(s) : \zeta'(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x-1}(\ln x - \psi(s))dx(14)$ where $\psi(s) = \Gamma'(s)/\Gamma(s)istheDigammafunction.$

4.2 Constructing the Coefficient Integrals

Substituting these integral forms into the analytical solutions derived in Section 2, we can merge the terms under a single integral. For $\alpha(s), thenumeratortermis\zeta(s)\ln P_b - \zeta'(s).Inintegralform :$ $\int_0^\infty \frac{x^{s-1}}{e^x-1}\left[\ln P_b - (\ln x - \psi(s))\right]dx(15)$ This leads to the strict integral representation:

$$\alpha(s) = \frac{1}{\mathcal{W}ab\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\left[\ln P_b + \psi(s) - \ln x\right] dx \tag{16}$$

Similarly for $\beta(s) : \beta(s) = \frac{1}{\mathcal{W}ab\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x-1}\left[\ln x - \psi(s) - \ln P_a\right]dx(17)$ Here, W$_{ab} = P_aP_b\ln(P_b/P_a)istheWronskia$
$theoreticvacuum(theBose-Einsteindistribution), independentoftheZetaseriessummation.$

5 The Barycentric Final Form

To finalize the theory, we solve the integral representations using Double Integration by Parts (IBP) and Cauchy's Residue Theorem. This process transforms the integral (defined on the real axis) into a sum of residues (on the imaginary axis), recovering the Reflection Formula components.

5.1 Double Integration by Parts (IBP)

We apply IBP twice to the kernel 1 $\frac{}{e^x-1toimproveconvergenceandrevealthecurvatureterm.I(s)=\frac{1}{s(s+1)}\int_0^\infty x^{s+1}\frac{d}{dx}\left(\frac{e^x}{(e^x-1)^2}\right)dx(18)}$
Evaluating the contour integral of this form sums the residues at the poles z$_n = 2\pi in.Thesumoftheseresiduesgener$
$\mathcal{R}(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)(19)$

5.2 Definitions of Spectral Components

We define two key components that govern the final structure: **1. The Global Magnitude (Reflection Kernel) R(s):** This represents the symmetric "echo" of the function from the critical line.

$$R(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{20}$$

2. The Differential Shift (Logarithmic Derivative) D(s): This represents the internal momentum of the function. It is derived from the derivative of the Reflection Kernel.

$$D(s) = \ln(2\pi) + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right) - \psi(1-s) - \frac{\zeta'(1-s)}{\zeta(1-s)} \tag{21}$$

Note that the term $\zeta'(1-s)\frac{}{\zeta(1-s)correspondstotheDirichletseriesoftheVonMangoldtfunction\Lambda(n),linkingtheshiftdirectlytothedistribu}$

5.3 The Barycentric Weights and Final Identity

We substitute R(s) and D(s) back into the coefficient equations. The complex derivatives collapse into a clean linear interpolation form. We define the **Barycentric Weights** W$_a(s)andW_b(s)basedonthepositionoftheDi$
$\ln P_b - \ln P_a.W_a(s) = \frac{\ln P_b - D(s)}{\Delta L}(22)$

$$W_b(s) = \frac{D(s) - \ln P_a}{\Delta L} \tag{23}$$

Observing that $\mathrm{W}_a(s)+W_b(s) = 1, wearriveatthefinalBarycentricIdentity : \boxed{\zeta(s) = \left(\frac{R(s)}{P_a}W_a(s)\right)P_a + \left(\frac{R(s)}{P_b}\right.}$

6 Conclusion

This research establishes that the Riemann Zeta function is strictly expressible as a dynamic linear interpolation between any two chosen prime numbers. The Two-Prime Basis system acts as a "Phase Mirror" on the critical line. The value of $\zeta(s) isdeterminednotbysumminginfiniteintegers, butbyaprecisetorqueba$
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